

SOME OSTROWSKI'S TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE s -CONVEX IN THE SECOND SENSE AND APPLICATIONS

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ABSTRACT. Some new inequalities of Ostrowski type for twice differentiable mappings whose derivatives in absolute value are s -convex in the second sense are given. Applications for special means are also provided.

1. INTRODUCTION

In 1938, Ostrowski proved the following integral inequality [12]:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$.*

Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For some applications of Ostrowski's inequality see ([1]-[4]) and for recent results and generalizations concerning Ostrowski's inequality see ([1]-[8]).

The class of s -convexity in the second sense is defined in the following way [9]: a function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and some fixed $s \in (0, 1]$. This class is usually denoted by K_s^2 .

In [10], Dragomir and Fitzpatrick proved the Hadamard's inequality for s -convex functions in the second sense:

Theorem 2. *Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$(1.1) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

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The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.1).

In [3], Cerone et.al. proved the following inequalities of Ostrowski type and Hadamard type, respectively.

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (a, b) and $f'' : (a, b) \rightarrow \mathbb{R}$ is bounded, i.e., $\|f''\|_\infty = \sup_{t \in (a, b)} |f''(t)| < \infty$. Then we have the inequality:*

$$(1.2) \quad \begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \left[\frac{1}{24} (b-a)^2 + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \|f''\|_\infty \\ & \leq \frac{(b-a)^2}{6} \|f''\|_\infty \end{aligned}$$

for all $x \in [a, b]$.

Corollary 1. *Under the above assumptions, we have the mid-point inequality:*

$$(1.3) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{24} \|f''\|_\infty$$

In this article, we establish new Ostrowski's type inequalities for s -convex functions in the second sense and using this results we note some applications to special means.

2. MAIN RESULTS

In order to establish our main results we need the following Lemma.

Lemma 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a, b]$, then*

$$(2.1) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \\ & = \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)a) dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)b) dt \end{aligned}$$

Proof. By integration by parts, we have the following identity

$$(2.2) \quad \begin{aligned} I_1 &= \int_0^1 t^2 f''(tx + (1-t)a) dt \\ &= \frac{t^2}{(x-a)} f'(tx + (1-t)a) \Big|_0^1 - \frac{2}{x-a} \int_0^1 t f'(tx + (1-t)a) dt \\ &= \frac{f'(x)}{(x-a)} - \frac{2}{x-a} \left[\frac{t}{(x-a)} f(tx + (1-t)a) \Big|_0^1 \right. \\ & \quad \left. - \frac{1}{x-a} \int_0^1 f(tx + (1-t)a) dt \right] \\ &= \frac{f'(x)}{(x-a)} - \frac{2f(x)}{(x-a)^2} + \frac{2}{(x-a)^2} \int_0^1 f(tx + (1-t)a) dt \end{aligned}$$

Using the change of the variable $u = tx + (1-t)a$ for $t \in [0, 1]$ and by multiplying the both sides (2.2) by $\frac{(x-a)^3}{2(b-a)}$, we obtain

$$(2.3) \quad \begin{aligned} & \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)a) dt \\ &= \frac{(x-a)^2 f'(x)}{2(b-a)} - \frac{(x-a)f(x)}{b-a} + \frac{1}{b-a} \int_a^x f(u) du \end{aligned}$$

Similarly, we observe that

$$(2.4) \quad \begin{aligned} & \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)b) dt \\ &= -\frac{(b-x)^2 f'(x)}{2(b-a)} - \frac{(b-x)f(x)}{b-a} + \frac{1}{b-a} \int_x^b f(u) du \end{aligned}$$

Thus, adding (2.3) and (2.4) we get the required identity (2.1). \square

The following result may be stated:

Theorem 4. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f''|$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$(2.5) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{1}{2(b-a)} \left\{ \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(a)|}{(s+1)(s+2)(s+3)} \right] (x-a)^3 \right. \\ & \quad \left. + \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(b)|}{(s+1)(s+2)(s+3)} \right] (b-x)^3 \right\} \end{aligned}$$

for each $x \in [a, b]$.

Proof. From Lemma 1 and since $|f''|$ is s -convex, then we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 [t^s |f''(x)| + (1-t)^s |f''(a)|] dt \\ & \quad + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 [t^s |f''(x)| + (1-t)^s |f''(b)|] dt \\ & = \frac{(x-a)^3}{2(b-a)} \int_0^1 (t^{s+2} |f''(x)| + t^2(1-t)^s |f''(a)|) dt \\ & \quad + \frac{(b-x)^3}{2(b-a)} \int_0^1 (t^{s+2} |f''(x)| + t^2(1-t)^s |f''(b)|) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{(x-a)^3}{2(b-a)} \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(a)|}{(s+1)(s+2)(s+3)} \right] \\
&\quad + \frac{(b-x)^3}{2(b-a)} \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(b)|}{(s+1)(s+2)(s+3)} \right] \\
&= \frac{1}{2(b-a)} \left\{ \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(a)|}{(s+1)(s+2)(s+3)} \right] (x-a)^3 \right. \\
&\quad \left. + \left[\frac{|f''(x)|}{s+3} + \frac{2|f''(b)|}{(s+1)(s+2)(s+3)} \right] (b-x)^3 \right\}
\end{aligned}$$

where we have used the fact that

$$\int_0^1 t^{s+2} dt = \frac{1}{s+3} \quad \text{and} \quad \int_0^1 t^2(1-t)^s dt = \frac{2}{(s+1)(s+2)(s+3)}.$$

This completes the proof. \square

Corollary 2. *We choose $|f''(x)| \leq M, M > 0$ in Theorem 4, then we have*

$$\begin{aligned}
(2.6) \quad &\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\
&\leq 3M \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right) \left[\frac{1}{24} (b-a)^2 + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \\
&\leq M \frac{(b-a)^2}{2} \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right).
\end{aligned}$$

Here, by simple computation shows that

$$(x-a)^3 + (b-x)^3 = (b-a) \left[\frac{(b-a)^2}{4} + 3 \left(x - \frac{a+b}{2} \right)^2 \right].$$

Remark 1. *If in Corollary 2 we choose $s = 1$, then we recapture the inequality (1.2).*

Corollary 3. *If in Corollary 2 we choose $x = \frac{a+b}{2}$, then we get the mid-point inequality*

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq M \frac{(b-a)^2}{2} \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right).$$

Theorem 5. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned}
(2.7) \quad &\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\
&\leq \frac{(x-a)^3}{2(b-a)} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left(\frac{|f''(x)|^q + |f''(a)|^q}{s+1} \right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{2(b-a)} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left(\frac{|f''(x)|^q + |f''(b)|^q}{s+1} \right)^{\frac{1}{q}}
\end{aligned}$$

for each $x \in [a, b]$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^3}{2(b-a)} \left(\int_0^1 t^{2p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^3}{2(b-a)} \left(\int_0^1 t^{2p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f''|^q$ is s -convex in the second sense, then we have

$$\begin{aligned} \int_0^1 |f''(tx + (1-t)a)|^q dt & \leq \int_0^1 [t^s |f''(x)|^q + (1-t)^s |f''(a)|^q] dt \\ & = \frac{|f''(x)|^q + |f''(a)|^q}{s+1} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 |f''(tx + (1-t)b)|^q dt & \leq \int_0^1 [t^s |f''(x)|^q + (1-t)^s |f''(b)|^q] dt \\ & = \frac{|f''(x)|^q + |f''(b)|^q}{s+1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{(x-a)^3}{2(b-a)} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left(\frac{|f''(x)|^q + |f''(a)|^q}{s+1} \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^3}{2(b-a)} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left(\frac{|f''(x)|^q + |f''(b)|^q}{s+1} \right)^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which is required. \square

Corollary 4. *Under the above assumptions we have the following inequality:*

$$\begin{aligned} (2.8) \quad & \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{3M}{(2p+1)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right]. \end{aligned}$$

This follows by Theorem 5, choosing $|f''(x)| \leq M$, $M > 0$.

Corollary 5. *With the assumptions in Corollary 4, one has the mid-point inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} M.$$

This follows by Corollary 4, choosing $x = \frac{a+b}{2}$.

Corollary 6. *With the assumptions in Corollary 4, one has the following perturbed trapezoid like inequality:*

$$\left| \int_a^b f(u)du - \frac{(b-a)}{2} [f(a) + f(b)] + \frac{(b-a)^2}{4} (f'(b) - f'(a)) \right| \leq \frac{(b-a)^3}{2(2p+1)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} M.$$

This follows using Corollary 4 with $x = a$, $x = b$, adding the results and using the triangle inequality for the modulus.

Theorem 6. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:*

$$(2.9) \quad \left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \leq \frac{(x-a)^3}{2(b-a)} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(a)|^q}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} + \frac{(b-x)^3}{2(b-a)} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(b)|^q}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}}$$

for each $x \in [a, b]$.

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ & \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^3}{2(b-a)} \left(\int_0^1 t^2 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^2 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^3}{2(b-a)} \left(\int_0^1 t^2 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^2 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f''|^q$ is s -convex in the second sense, we have

$$\begin{aligned} \int_0^1 t^2 |f''(tx + (1-t)a)|^q dt & \leq \int_0^1 [t^{s+2} |f''(x)|^q + t^2(1-t)^s |f''(a)|^q] dt \\ & = \frac{|f''(x)|^q}{s+3} + \frac{2|f''(a)|^q}{(s+1)(s+2)(s+3)} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 t^2 |f''(tx + (1-t)b)|^q dt &\leq \int_0^1 [t^{s+2} |f''(x)|^q + t^2(1-t)^s |f''(b)|^q] dt \\ &= \frac{|f''(x)|^q}{s+3} + \frac{2|f''(b)|^q}{(s+1)(s+2)(s+3)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ &\leq \frac{(x-a)^3}{2(b-a)} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(a)|^q}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^3}{2(b-a)} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(b)|^q}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}}. \end{aligned}$$

□

Corollary 7. *Under the above assumptions we have the following inequality*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ &\leq M \left(\frac{3(s^2 + 3s + 4)}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2}\right)^2 \right]. \end{aligned}$$

This follows by Theorem 6, choosing $|f''(x)| \leq M$, $M > 0$.

Corollary 8. *With the assumptions in Corollary 7, one has the mid-point inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq M \left(\frac{3(s^2 + 3s + 4)}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} \frac{(b-a)^2}{24}.$$

This follows by Corollary 7, choosing $x = \frac{a+b}{2}$.

Remark 2. *If in Corollary 8 we choose $s = 1$ and $q = 1$, then we have the following inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq M \frac{(b-a)^2}{24}$$

which is the inequality (1.3).

Corollary 9. *With the assumptions in Corollary 7, one has the following perturbed trapezoid like inequality:*

$$\begin{aligned} &\left| \int_a^b f(u) du - \frac{(b-a)}{2} [f(a) + f(b)] + \frac{(b-a)^2}{4} (f'(b) - f'(a)) \right| \\ &\leq \frac{(b-a)^3}{6} \left(\frac{3(s^2 + 3s + 4)}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} M. \end{aligned}$$

This follows using Corollary 7 with $x = a$, $x = b$, adding the results and using the triangle inequality for the modulus.

Remark 3. All of the above inequalities obviously hold for convex functions. Simply choose $s = 1$ in each of those results to get desired results.

The following result holds for s -concave.

Theorem 7. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is s -concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$(2.10) \quad \left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ \leq \frac{2^{(s-1)/q}}{(2p+1)^{1/p} (b-a)} \left(\frac{(x-a)^3 \left| f''\left(\frac{x+a}{2}\right) \right| + (b-x)^3 \left| f''\left(\frac{b+x}{2}\right) \right|}{2} \right)$$

for each $x \in [a, b]$.

Proof. Suppose that $q > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt \\ \leq \frac{(x-a)^3}{2(b-a)} \left(\int_0^1 t^{2p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ + \frac{(b-x)^3}{2(b-a)} \left(\int_0^1 t^{2p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s -concave in the second sense, using the inequality (1.1)

$$(2.11) \quad \int_0^1 |f''(tx + (1-t)a)|^q dt \leq 2^{s-1} \left| f''\left(\frac{x+a}{2}\right) \right|^q$$

and

$$(2.12) \quad \int_0^1 |f''(tx + (1-t)b)|^q dt \leq 2^{s-1} \left| f''\left(\frac{b+x}{2}\right) \right|^q.$$

A combination of (2.11) and (2.12) inequalities, we get

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left(x - \frac{a+b}{2} \right) f'(x) \right| \\ \leq \frac{2^{(s-1)/q}}{(2p+1)^{1/p} (b-a)} \left(\frac{(x-a)^3 \left| f''\left(\frac{x+a}{2}\right) \right| + (b-x)^3 \left| f''\left(\frac{b+x}{2}\right) \right|}{2} \right).$$

This completes the proof. \square

Corollary 10. *If in (2.10), we choose $x = \frac{a+b}{2}$, then we have*

$$(2.13) \quad \left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{2^{(s-1)/q} (b-a)^2}{16 (2p+1)^{1/p}} \left[\left| f''\left(\frac{3a+b}{4}\right) \right| + \left| f''\left(\frac{a+3b}{4}\right) \right| \right].$$

For instance, if $s = 1$, then we have

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{16 (2p+1)^{1/p}} \left[\left| f''\left(\frac{3a+b}{4}\right) \right| + \left| f''\left(\frac{a+3b}{4}\right) \right| \right].$$

3. APPLICATIONS TO SOME SPECIAL MEANS

Let us recall the following special means:

(1) The arithmetic mean:

$$A = A(x, y) := \frac{x+y}{2}, \quad x, y \geq 0;$$

(2) The Identric mean:

$$I = I(x, y) := \begin{cases} x & \text{if } x = y \\ \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{\frac{1}{y-x}} & \text{if } x \neq y \end{cases}, \quad x, y > 0;$$

(c) The generalized log-mean:

$$L_p = L_p(a, b) := \begin{cases} x & \text{if } x = y \\ \left[\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)} \right]^{\frac{1}{p}} & \text{if } x \neq y \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad x, y > 0.$$

The following simple relationship is well known in the literature

$$I \leq A.$$

It is known that L_p is monotonic nondecreasing in $p \in \mathbb{R}$ with $L_o := I$.

Now, using the results of Section 2, we give some applications to special means of positive real numbers.

In [11], the following example is given: Let $0 < s < 1$ and $u, v, w \in \mathbb{R}$. We define a function $f : [0, \infty) \rightarrow \mathbb{R}$

$$f(t) = \begin{cases} u & \text{if } t = 0 \\ vt^s + w & \text{if } t > 0. \end{cases}$$

If $v \geq 0$ and $0 \leq w \leq u$, then $f \in K_s^2$. Hence, for $u = w = 0$, $v = 1$, we have $f : [0, 1] \rightarrow [0, 1]$, $f(t) = t^s$, $f \in K_s^2$.

Proposition 1. *Let $0 < a < b$ and $s \in (0, 1)$. Then we have the results:*

$$(3.1) \quad \begin{aligned} & |L_s^s(a, b) - x^s + s(x - A)x^{s-1}| \\ & \leq 3M \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right) \left[\frac{1}{24} (b-a)^2 + \frac{1}{2} (x-A)^2 \right] \\ & \leq M \frac{(b-a)^2}{2} \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right) \end{aligned}$$

for all $x \in [a, b]$. If in the first inequality of (3.1) we choose $x = A$, we get

$$|L_s^s(a, b) - A^s| \leq \frac{M(b-a)^2}{8} \left(\frac{s^2 + 3s + 4}{(s+1)(s+2)(s+3)} \right).$$

Proof. The inequality of (3.1) follows from (2.6) applied to the s -convex function in the second sense $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^s$. The details are omitted. \square

Proposition 2. Let $0 < a < b$, $q > 1$ and $s \in (0, 1)$. Then we have the results:

$$(3.2) \quad \begin{aligned} & |L_s^s(a, b) - x^s + s(x - A)x^{s-1}| \\ & \leq \frac{3M}{(2p+1)^{1/p}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}(x-A)^2 \right] \end{aligned}$$

for all $x \in [a, b]$. If in the first inequality of (3.2) we choose $x = A$, we get

$$|L_s^s(a, b) - A^s| \leq \frac{M}{8(2p+1)^{1/p}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} (b-a)^2.$$

Proof. The proof of (3.2) is similar to that of (3.1), using the inequality (2.8). \square

Proposition 3. Let $0 < a < b$, $q > 1$ and $s \in (0, 1)$. Then we have the results:

$$(3.3) \quad \begin{aligned} & |L_s^s(a, b) - x^s + s(x - A)x^{s-1}| \\ & \leq M \left(\frac{3(s^2 + 3s + 4)}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}(x-A)^2 \right] \end{aligned}$$

for all $x \in [a, b]$. If in (3.3) we choose $x = A$, we get

$$|L_s^s(a, b) - A^s| \leq \frac{M}{24} \left(\frac{3(s^2 + 3s + 4)}{(s+1)(s+2)(s+3)} \right)^{\frac{1}{q}} (b-a)^2.$$

Proof. The proof is similar to that of Proposition 1, using Corollary 7. \square

Proposition 4. Let $0 < a < b$, $p > 1$ and $s \in (0, 1)$. Then we have the result:

$$|\ln I - \ln A| \leq \frac{2^{(s-1)/q}(b-a)^2}{(2p+1)^{1/p}} \left[-\frac{1}{(3a+b)^2} - \frac{1}{(a+3b)^2} \right].$$

Proof. The inequality follows from (2.13) applied to the concave function in the second sense $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = \ln x$. The details are omitted. \square

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